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1997 J. Phys. A: Math. Gen. 30 5159

(http://iopscience.iop.org/0305-4470/30/14/025)

J. Phys. A: Math. Gen. **30** (1997) 5159–5177. Printed in the UK PII: S0305-4470(97)80859-1

Second-order second degree Painlevé equations related with Painlevé I, II, III equations

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Received 30 December 1996

Abstract. The algorithmic method introduced by Fokas and Ablowitz to investigate the transformation properties of Painlevé equations is used to obtain a one-to-one correspondence between the Painlevé I, II and III equations and certain second-order second degree equations of Painlevé type.

1. Introduction

Second-order and first degree equations

$$y'' = F(z, y, y') \tag{1.1}$$

where F is rational in y', algebraic in y and locally analytic in z with the property that the only movable singularities are poles, that is, the Painlevé property, were classified at the turn of the century by Painlevé and his school [22, 15, 17]. Within the Möbius transformation, they found 50 such equations. Among all these equations, six of them are irreducible and define classical Painlevé transcendents, PI, PII, ..., PVI. The remaining 44 equations are either solvable in terms of the known functions or can be transformed into one of the six equations.

Besides the physical importance, the Painlevé equations possess a rich internal structure. Some of these properties can be summarized as follows. (i) For a certain choice of parameters, PII–PVI admit a one-parameter family of solutions which are either rational or expressible in terms of the classical transcendental functions. For example, PII admits a one-parameter family of solutions expressible in terms of Airy functions [9]. (ii) There are transformations (Bäcklund or Schlesinger) associated with PII–PVI, these transformations map the solution of a given Painlevé equation to the solution of the same equation but with different values of parameters [11, 19, 20]. (iii) PI–PV can be obtained from PVI by the process of contraction [17]. It is possible to obtain the associated transformations for PII–PV from the transformation for PVI. (iv) They can be obtained as the similarity reduction of the nonlinear partial differential equations solvable by inverse scattering transform (IST). Since the work of Kowalevskaya that was the first connection between the integrability and the Painlevé property. (v) PI–PVI can be considered as the isomonodromic conditions of a suitable linear system of ordinary differential equations with rational coefficients possessing both regular and irregular singularities [18]. Moreover, the initial value problem of PI–PVI can be studied by using the inverse monodromy transform (IMT) [12, 13, 21].

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The Riccati equation is the only example of the first-order first degree equation which has the Painlevé property. Before the work of Painlevé and his school Fuchs [14, 17] considered the equation of the form

$$F(z, y, y') = 0 \tag{1.2}$$

where F is polynomial in y and y' and locally analytic in z, such that the movable branch points are absent, that is, the generalization of the Riccati equation. Briot and Bouquet [17] considered the subcase of (1.2), that is, first-order binomial equations of degree $m \in \mathbb{Z}_+$:

$$(y')^m + F(z, y) = 0 \tag{1.3}$$

where $F(z, y)$ is a polynomial of degree at most $2m$ in y. It was found that there are six types of equation of the form (1.3). But, all these equations are either reducible to a linear equation or solvable by means of elliptic functions [17]. Second-order binomial-type equations of degree $m \geqslant 3$

$$(y'')^m + F(z, y, y') = 0 \tag{1.4}$$

where F is polynomial in y and y' and locally analytic in z, was considered by Cosgrove [4]. It was found that there are nine classes. Only two of these classes have arbitrary degree m and the others have degree three, four and six. As in the case of first-order binomial-type equations, all these nine classes are solvable in terms of the first, second and fourth Painlevé transcendents, elliptic functions or by quadratures. Chazy [3], Garnier [16] and Bureau [1] considered the third-order differential equations possessing the Painlevé property of the following form

$$y''' = F(z, y, y', y'') \tag{1.5}$$

where F is assumed to be rational in y, y', y'' and locally analytic in z. But, in [1] the special form of $F(z, y, y', y'')$

$$F(z, y, y', y'') = f_1(z, y)y'' + f_2(z, y)(y')^2 + f_3(z, y)y' + f_4(z, y) \tag{1.6}$$

where $f_k(z, y)$ are polynomials in y of degree k with analytic coefficients in z was considered. In this class no new Painlevé transcendents were discovered and all of them are solvable either in terms of the known functions or one of six Painlevé transcendents.

Second-order second degree Painlevé-type equations of the following form

$$(y'')^2 = E(z, y, y')y'' + F(z, y, y') \tag{1.7}$$

where E and F are assumed to rational in y, y' and locally analytic in z was the subject of the articles [2, 8]. In [2] the special case of (1.7)

$$y'' = M(z, y, y') + \sqrt{N(z, y, y')} \tag{1.8}$$

was considered, where M and N are polynomials of degree two and four respectively in y', rational in y and locally analytic in z. Also, in this classification, no new Painlevé transcendents were found. In [8], the special form, $E = 0$ and hence F is polynomial in y and y' of (1.7) was considered and that six distinct class of equations were obtained by using the so-called α-method. These classes were denoted by SD − I, ... , SD − VI and are solvable in terms of the classical Painlevé transcendents (PI, ... , PVI), elliptic functions or solutions of the linear equations.

Second-order second degree equations of Painlevé type appear in physics [5–7]. Moreover, second degree equations are also important in determining transformation properties of the Painlevé equations [10, 11]. In [11], the aim was to develop an algorithmic method to investigate the transformation properties of the Painlevé equations. But, certain new second degree equations of Painlevé type related with PIII and PVI were also discussed. By using the same notation, the algorithm introduced in [11] can be summarized as follows. Let $v(z)$ be a solution of any of the fifty Painlevé equations, as listed by Gambier [15] and Ince [17], each of which takes the form

$$v'' = P_1(v')^2 + P_2v' + P_3 \tag{1.9}$$

where P_1, P_2, P_3 are functions of v, z and a set of parameters $\boldsymbol{\alpha}$. The transformation, i.e. Lie-point discrete symmetry which preserves the Painlevé property of (1.9) of the form $u(z; \hat{\boldsymbol{\alpha}}) = F(v(z; \boldsymbol{\alpha}), z)$ is the Möbius transformation

$$u(z; \hat{\boldsymbol{\alpha}}) = \frac{a_1(z)v + a_2(z)}{a_3(z)v + a_4(z)} \tag{1.10}$$

where $v(z, \boldsymbol{\alpha})$ solves (1.9) with the set of parameters $\boldsymbol{\alpha}$ and $u(z; \hat{\boldsymbol{\alpha}})$ solves (1.9) with the set of parameters $\hat{\boldsymbol{\alpha}}$. Lie-point discrete symmetry (1.10) can be generalized by involving the $v'(z; \alpha)$, i.e. the transformation of the form $u(z; \hat{\boldsymbol{\alpha}}) = F(v'(z; \alpha), v(z, \alpha), z)$. The only transformation which contains v' linearly is the one involving the Riccati equation, i.e.

$$u(z, \hat{\boldsymbol{\alpha}}) = \frac{v' + av^2 + bv + c}{dv^2 + ev + f} \tag{1.11}$$

where a, b, c, d, e, f are functions of z only. The aim is to find a, b, c, d, e, f such that (1.11) defines a one-to-one invertible map between solutions v of (1.9) and solutions u of some second-order equation of the Painlevé type. Let

$$J = dv^2 + ev + f \qquad Y = av^2 + bv + c \tag{1.12}$$

then differentiating (1.11) and using (1.9) to replace v'' and (1.11) to replace v', one obtains:

$$\begin{aligned} Ju' = {} & [P_1J^2 - 2dJv - eJ]u^2 + [-2P_1JY + P_2J + 2avJ \\ & + bJ + 2dvY + eY - (d'v^2 + e'v + f')]u + [P_1Y^2 - P_2Y \\ & + P_3 - 2avY - bY + a'v^2 + b'v + c']. \end{aligned} \tag{1.13}$$

There are two distinct cases.

(I) Find $a, \ldots, f$ such that (1.13) reduces to a linear equation for v,

$$A(u', u, z)v + B(u', u, z) = 0. \tag{1.14}$$

Having determined $a, \ldots, f$ upon substitution of $v = -B/A$ into (1.11) one can obtain the equation for u, which will be one of the fifty Painlevé equations.

(II) Find $a, \ldots, f$ such that (1.13) reduces to a quadratic equation for v,

$$A(u', u, z)v^2 + B(u', u, z)v + C(u', u, z) = 0. \tag{1.15}$$

Then (1.11) yields an equation for u which is quadratic in the second derivative.

As mentioned before in [11] the aim is to obtain the transformation properties of PII–PVI. Hence, the case I for PII–PV, and case II for PVI was investigated.

In this article, we investigate the transformation of type II to obtain the one-to-one correspondence between PI, PII, PIII and the second-order second degree Painlevé-type equations. Some of the second degree equations related with PI–PIII were obtained in [2, 8] but most of them have not been considered in literature. Instead of having the transformation of the form (1.11) which is linear in v', one may use the appropriate transformations related to

$$(v')^m + \sum_{j=1}^{m} P_j(z, v)(v')^{m-j} = 0 \qquad m > 1$$

where $P_j(z, v)$ is a polynomial in v, which satisfies the Fuchs theorem concerning the absence of movable critical points [14, 17]. This type of transformations and the transformations of type II for PIV–PVI will be published elsewhere.

2. Painlevé I

Let $v(z)$ be a solution of PI

$$v'' = 6v^2 + z. \tag{2.1}$$

Then, for PI equation (1.13) takes the form of

$$\begin{aligned}&[2d^2u^2 - 4adu + 2a^2]v^3 + [du' + 3deu^2 + (d' - 3ae - 3bd)u - (a' - 3ab + 6)]v^2 \\ &\quad + [eu' + (2df + e^2)u^2 + (e' - 2af - 2be - 2cd)u - (b' - b^2 - 2ac)]v \\ &\quad + [fu' + efu^2 + (f' - bf - ec)u - (c' - bc + z)] = 0.\end{aligned} \tag{2.2}$$

Now, the aim is to choose $a, b, \dots, f$ in such a way that (2.2) becomes a quadratic equation for v. There are two cases: either the coefficient of v^3 is zero or not.

Case I. $2d^2u^2 - 4adu + 2a^2 = 0$. In this case the only possibility is $a = d = 0$, and one has to consider the two cases separately (i) $e = 0$ and (ii) $e \neq 0$.

Case I.i. $e = 0$. One can always absorb c and f in u by a proper Möbius transformation, and hence, without loss of generality, one sets $c = 0$, and $f = 1$. Then equation (2.2) takes the following form,

$$6v^2 + (b' - b^2)v - (u' - bu - z) = 0. \tag{2.3}$$

The procedure discussed in the introduction yields the following second-order second degree Painlevé-type equation for $u(z)$

$$\begin{aligned}&[u'' + bu' - (b' + 2b^2)u + \tfrac{1}{12}(b' - b^2)(b'' - bb' - b^3) - 2zb - 1]^2 \\ &\quad = [u + \tfrac{1}{12}(b'' - bb' - b^3)]^2[24u' - 24bu + (b' - b^2)^2 - 24z]\end{aligned} \tag{2.4}$$

and there exist the following one-to-one correspondence between solutions $v(z)$ and $u(z)$

$$u = v' + bv \qquad v = \frac{u'' + bu' - (2b' + b^2)u - 2zb - 1}{12u + b'' - bb' - b^3}. \tag{2.5}$$

The change of variable $u(z) = p(x)y(x) + q(x)$, $z = z(x)$ where

$$\begin{aligned}&f(x) = c_1x + c_2 \\ &R(x) = \exp\left(-5\int b(z)\,\mathrm{d}z\right) \\ &z = c_4^{-4/5}\left(c_4\int f^{-2}R^{-3/5}\,\mathrm{d}x + c_5\right) \\ &p(x) = \tfrac{1}{24}c_4^{-3/5}f^{-1}R^{-1/5} \\ &q(x) = \tfrac{1}{24}c_4^{3/5}R^{1/5}\left(\int\left\{f^6\left[-\frac{1}{5}\ddot{R} + \frac{1}{25}R^{-1}\dot{R}^2 - \frac{2c_1}{5f}\dot{R}\right]^2\right.\right. \\ &\qquad\left.\left. -24c_4^{4/5}f^{-2}R^{-2/5}z(x) + c_3f^{-2}\right\}\mathrm{d}x + c_6\right)\end{aligned} \tag{2.6}$$

c_j, $j = 1, 2, \dots, 6$ are constants and $\dot{R} = \frac{\mathrm{d}R}{\mathrm{d}x}$, transforms (2.4) into the following form,

$$\ddot{y}^2 = [A(x)y + B(x)]^2[c_1(x\dot{y} - y) + c_2\dot{y} + c_3] \tag{2.7}$$

where $A(x)$ and $B(x)$ are given in terms of $f(x)$ and $R(x)$. Equation (2.7) was first obtained by Cosgrove and Scoufis [8] and labelled as SD-V.A.

Case I.ii. $e \neq 0$. Without loss of generality one can set $b = 0$ and $e = 1$. Hence, equations (1.11) and (2.2) become

$$u = \frac{v' + c}{v + f} \qquad Av^2 + Bv + C = 0 \tag{2.8}$$

respectively, where

$$\begin{aligned} &A = 6 \qquad B = -(u' + u^2) \\ &C = -(fu' + fu^2 - a_1 u - a_0 + 6f^2) \\ &a_1 = c - f' \qquad a_0 = c' + 6f^2 + z. \end{aligned} \tag{2.9}$$

The discriminant Δ of the second equation of (2.8) is

$$\Delta = (u' + u^2 + 12f)^2 - 24(a_1 u + a_0). \tag{2.10}$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero, then the first equation of (2.8) and

$$v = -\frac{fu'' + (fu - 2a_1)u' - fu^3 + a_1 u^2 - (a_1' - 2a_0 + 12f^2)u - c'' - 1}{u'' + uu' - u^3 - 12fu - 12c} \tag{2.11}$$

define a one-to-one correspondence between a solution $v(z)$ of PI and a solution $u(z)$ of the following second degree equation

$$\begin{aligned} &[2(a_1 u + a_0)u'' - 2a_1 u'^2 + R_2(u)u' - Q_4(u)]^2 \\ &\qquad = [2a_1 u' - a_1 u^2 + (a_1' - 2a_0)u + (a_0' + 12fa_1)]^2 \Delta \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} R_2(u) &= a_1 u^2 - (a_1' - 4a_0)u - (a_0' + 36fa_1) \\ Q_4(u) &= a_1 u^4 + a_1' u^3 + (a_0' + 24fa_1)u^2 + 12(fa_1' - 2f'a_1 - 2a_1^2)u \\ &\quad + 12(fa_0' - 2f'a_0 + 12f^2 a_1 - 2a_0 a_1). \end{aligned} \tag{2.13}$$

Note that if $a_1 = f = 0$, then $y = -u$ solves the following equation

$$y'' - 2yy' = \frac{1}{2z}(y' - y^2) + \left(y + \frac{1}{2z}\right)\sqrt{(y' - y^2)^2 - 24z}. \tag{2.14}$$

The second-order second degree Painlevé-type equation for $y(z)$ was first obtained by Bureau [2]. If Δ is a complete square, that is $a_0 = a_1 = 0$ then u satisfies PX in [17 p 334].

Case II. $2d^2u^2 - 4adu + 2a^2 \neq 0$. In this case equation (2.2) can be written as

$$(v + h)(Av^2 + Bv + C) = 0 \tag{2.15}$$

where

$$\begin{aligned} A &= 2d^2u^2 - 4adu + 2a^2 \\ B &= du' + d(3e - 2dh)u^2 + (d' - 3ae - 3bd + 4adh)u - (a' - 3ab + 2a^2h + 6) \\ C &= (e - dh)u' + (e^2 + 2df - 3deh + 2d^2h^2)u^2 \\ &\quad + (e' - hd' - 2af - 2be - 2cd + 3aeh + 3bdh - 4adh^2)u \\ &\quad - (b' - b^2 - ha' - 2ac + 3abh - 6h - 2a^2h^2) \end{aligned} \tag{2.16}$$

and h is a function of z. $f = h(e - dh)$ and b, c, d, e satisfy the following equations

$$\begin{aligned} &(e - 2dh)(h' + bh - ah^2 - c) = 0 \\ &c' - bc + z = h(b' - ha' - b^2 - 2ac + 3abh - 6h - 2a^2h^2). \end{aligned} \tag{2.17}$$

One has to distinguish two cases: (i) $d = 0$ and (ii) $d \neq 0$.

Case II.i. $d = 0$. When $d = 0$, without loss of generality, one can choose $b = 0$ and $e = 1$, then equations (1.11) and (2.2) take the following forms

$$u = \frac{v' + av^2 + c}{v + f} \qquad Av^2 + Bv + C = 0 \tag{2.18}$$

respectively, where

$$\begin{aligned} &A = 2a^2 \qquad B = -(3au + a' + 2a^2 f + 6) \\ &C = u' + u^2 + afu + f(a' + 2a^2 f + 6) + 2ac. \end{aligned} \tag{2.19}$$

Clearly, a should be different than zero, then (2.17) and $f = h$ yield

$$c = f' - af^2 \qquad f'' + 6f^2 + z = 0. \tag{2.20}$$

Then, for these choices u satisfies the following second degree equation of Painlevé type

$$[8a^3u'' + 2a^2(3au - 7a' + 6a^2 f + 6)u' - Q_3(u)]^2 = [2a^2u' - R_2(u)]^2\Delta \tag{2.21}$$

where

$$\begin{aligned} &\Delta = -(8a^2u' - a^2u^2 - 2aa_1u - a_0) \\ &Q_3(u) = a^3u^3 + a^2(5a' + 6a^2 f + 42)u^2 + a[2aa_1' - 2a_1(2a' - 2a^2 f - 6) + a_0]u \\ &\qquad\quad +aa_0' - a_0(3a' - 2a^2 f - 6) \\ &R_2(u) = a^2u^2 + 2a(a' + 2a^2 f - 12)u + 2a'' - 3a'^2 - 12a' + 4ca^3 + 36 \\ &a_1 = 3a' + 2a^2 f + 18 \\ &a_0 = a'^2 - 4(a^2 f - 3)a' - 4a^2(4ac + 3a^2 f^2 + 6f) + 36. \end{aligned} \tag{2.22}$$

Case II.ii. $d \neq 0$. Without loss of generality, one can set $a = 0, d = 1$. Then $f = h(e-h)$ and the first equation of (2.17) gives

$$(e - 2h)(h' + bh - c) = 0. \tag{2.23}$$

If $e = 2h$, then $f = h^2$ and equations (1.11) and (2.2) become

$$u = \frac{v' + bv + c}{(v + h)^2} \qquad Av^2 + Bv + C = 0 \tag{2.24}$$

respectively, where

$$\begin{aligned} &A = 2u^2 \qquad B = u' + 4hu^2 - 3bu - 6 \\ &C = hu' + 2h^2u^2 + (a_1 - 3bh)u + a_0 - 6h \\ &a_1 = 2(h' + bh - c) \qquad a_0 = -(b' - b^2 - 12h) \\ &c' - bc + z + h(a_0 - 6h) = 0. \end{aligned} \tag{2.25}$$

The discriminant Δ of the second equation of (2.24) is

$$\Delta = (u' - 3bu - 6)^2 - 8u^2(a_1u + a_0). \tag{2.26}$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero, then u satisfies the following second degree equation

$$[4u(a_1u+a_0)u''-3(2a_1u+a_0)u'^2-R_2(u)u'+Q_3(u)]^2$$
$$=[3a_0u'-2(a_1'-ba_1)u^2-(2a_0'+ba_0-12a_1)u+6a_0]^2\Delta \tag{2.27}$$

where

$$R_2(u)=2[(a_1'-3ba_1)u^2+(a_0'+3ba_0-18a_1)u-6a_0]$$
$$Q_3(u)=2[3ba_1'+2a_1(a_0-36h-3b^2)]u^3 \tag{2.28}$$
$$+[12a_1'+3b(2a_0'-ba_0)+4a_0(a_0-36h)]u^2+12(a_0'+3ba_0)u+36a_0.$$

If $e\neq 2h$, then $c=h'+bh$, $f=h(e-h)$ and equations (1.11) and (2.2) become

$$u=\frac{v'+bv+c}{(v+h)(v+e-h)} \qquad Av^2+Bv+C=0 \tag{2.29}$$

respectively, where

$$\begin{aligned}&A=2u^2 \qquad B=u'+(3e-2h)u^2-3bu-6\\&C=(e-h)u'+e(e-h)u^2+(a_1-3be+3bh)u+a_0-6(e-h)\\&a_1=e'-2h'+b(e-2h) \qquad a_0=-(b'-b^2-6e)\\&h''+6h^2+z=0.\end{aligned} \tag{2.30}$$

The discriminant Δ of the second equation of (2.29) is

$$\Delta=[u'-(e-2h)u^2-3bu-6]^2-8u^2(a_1u+a_0). \tag{2.31}$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero, then u satisfies the following second degree equation

$$[4u(a_1u+a_0)u''-(7a_1u+3a_0)u'^2-F_2(u)u'-Q_5(u)]^2=[(a_1u-3a_0)u'+R_3(u)]^2\Delta \tag{2.32}$$

where

$$F_2(u)=[a_1'-6ba_1+3a_0(e-2h)]u^2+(a_0'+3ba_0-24a_1)u-6a_0$$
$$\begin{aligned}Q_5(u)=&\,a_1(e-2h)^2u^5-[2(e-2h)(a_1'-ba_1)+4a_1^2-(e-2h)^2a_0]u^4\\&-[3b(2a_1'-7ba_1)+4a_1(2a_0+6h-21e)+2(e-2h)(a_0'+5ba_0)]u^3\\&-[12(a_1'-3ba_1)+3b(2a_0'-ba_0)+4a_0(a_0-15e-6h)]u^2\\&-12(a_0'+3ba_0-3a_1)u-36a_0\end{aligned} \tag{2.33}$$
$$R_3(u)=a_1(e-2h)u^3+[2a_1'-5ba_1-3a_0(e-2h)]u^2+(2a_0'+ba_0-18a_1)u-6a_0.$$

If Δ is a complete square, that is, $a_1=a_0=0$, then $w=6/u$ solves PXXVIII [17, p 340].

3. Painlevé II

In this section we consider the equation PII. Let $v(z)$ be a solution of PII

$$v''=2v^3+zv+\alpha. \tag{3.1}$$

One finds that $P_1 = P_2 = 0$ and $P_3 = 2v^3 + zv + \alpha$ by comparing (3.1) with (1.9). Then equation (1.13) becomes

$$[2d^2u^2 - 4adu + 2a^2 - 2]v^3 + [du' + 3deu^2 + (d' - 3ae - 3bd)u - (a' - 3ab)]v^2 \\ +[eu' + (2df + e^2)u^2 + (e' - 2af - 2be - 2cd)u - (b' - b^2 - 2ac + z)]v \\ +[fu' + efu^2 + (f' - bf - ec)u - (c' - bc + \alpha)] = 0. \qquad (3.2)$$

To reduce (3.2) to a quadratic equation for v, there are two cases depending on whether the coefficient of v^3 is zero or not.

Case I. $2d^2u^2 - 4adu + 2a^2 - 2 = 0$. This implies that $d = 0$, $a^2 = 1$. One has to consider the two cases: (i) $e = 0$, and (ii) $e \neq 0$ separately.

Case I.i. $e = 0$. With a proper Möbius transformation, one can choose $c = 0$, and $f = 1$. Then equations (1.11) and (3.2) take the form of

$$u = v' + av^2 + bv \qquad Av^2 + Bv + C = 0 \qquad (3.3)$$

respectively, where

$$A = -3ab \qquad B = 2au + b_0 \\ C = -(u' - bu - \alpha) \qquad b_0 = b' - b^2 + z. \qquad (3.4)$$

When $b \neq 0$, $u(z)$ satisfies the following second-order second degree Painlevé-type equation:

$$[18b^2u'' + 6b(2au - 2b_0 + 3z)u' - Q_3(u)]^2 \\ = [4u^2 - 2a(b_0 + 6b^2 - 3z)u + 3bb_0' - 2b_0^2 + 3zb_0 + 6a\alpha b]^2\Delta \qquad (3.5)$$

where

$$\Delta = -[12abu' - 4u^2 - 4a(b' + 2b^2 + z)u - b_0^2 - 12\alpha ab] \\ Q_3(u) = 8u^3 + 12a(2b^2 + z)u^2 + 6(bb_0' - b_0^2 + 2b^2b_0 + 2zb_0 + 6b^4 + 4a\alpha b)u \\ +3abb_0b_0' - 2ab_0^3 + 3azb_0^2 - 6\alpha bb_0 + 18\alpha b(b^2 + z). \qquad (3.6)$$

When $b = 0$ the discriminant Δ is a complete square, u is a solution of PXXXIV in [17, p 340].

Case I.ii. $e \neq 0$. Without loss of generality, one can choose $b = 0$ and $e = 1$. Hence equations (1.11) and (3.2) become

$$u = \frac{v' + av^2 + c}{v + f} \qquad Av^2 + Bv + C = 0 \qquad (3.7)$$

respectively, where

$$A = 3au \qquad B = -(u' + u^2 - 2afu + b_0) \\ C = -[fu' + fu^2 + (a_1 + af^2)u + a_0 + fb_0] \qquad (3.8) \\ a_1 = f' - af^2 - c \qquad a_0 = -(c' + 2afc - zf + \alpha) \qquad b_0 = 2ac - z.$$

The discriminant Δ of the second equation of (3.7) is

$$\Delta = (u' + u^2 + 4afu + 2ac - z)^2 + 12au(a_1u + a_0). \qquad (3.9)$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero. Then u satisfies the following second degree equation

$$[6u(a_1u + a_0)u'' - 2(4a_1u + a_0)u'^2 + F_3(u)u' - Q_5(u)]^2 = [2(2a_1u - a_0)u' - R_3(u)]^2\Delta \qquad (3.10)$$

where

$$\begin{aligned}
F_3(u) &= 2a_1u^3 - (3a_1' - 11a_0 + 16afa_1)u^2 - (3a_0' - 20afa_0 + 10a_1b_0)u - a_0b_0 \\
Q_5(u) &= 2a_1u^5 + (3a_1' - a_0 + 16afa_1)u^4 + [3a_0' + 12afa_1' + 4(2f^2 - 4ac - z)a_1 \\
&\quad -8afa_0]u^3 + [b_0(3a_1' + 28afa_1) + 12afa_0' + 6(2a\alpha + 1)a_1 \\
&\quad -2a_0(20f^2 + 14ac - z)]u^2 + [b_0(3a_0' + 4afa_0) \\
&\quad +6a_0(2a\alpha + 1) + 2a_1b_0^2]u - a_0b_0^2 \\
R_3(u) &= 2a_1u^3 - (3a_1' + 4afa_1 - 5a_0)u^2 - (3a_0' - 8afa_0 + 4a_1b_0)u - a_0b_0.
\end{aligned} \tag{3.11}$$

If Δ is a complete square, that is, $a_1 = a_0 = 0$, then u satisfies PXXXV [17, p 340].

Case II. $2d^2u^2 - 4adu + 2a^2 - 2 \neq 0$. In this case (3.2) can be written as

$$(v + h)(Av^2 + Bv + C) = 0 \tag{3.12}$$

where

$$\begin{aligned}
A &= 2d^2u^2 - 4adu + 2a^2 - 2 \\
B &= du' + d(3e - 2dh)u^2 + (d' - 3ae - 3bd + 4adh)u - (a' - 3ab + 2a^2h - 2h) \\
C &= (e - dh)u' + (e^2 + 2df - 3deh + 2d^2h^2)u^2 \\
&\quad +(e' - hd' - 2af - 2be - 2cd + 3aeh + 3bdh - 4adh^2)u \\
&\quad -(b' - b^2 - ha' - 2ac + 3abh + 2h^2 - 2a^2h^2 + z)
\end{aligned} \tag{3.13}$$

h is a function of z, $f = h(e - dh)$, and b, c, d, e satisfy the following equations

$$\begin{aligned}
&(e - 2dh)(h' - ah^2 + bh - c) = 0 \\
&c' - bc + \alpha = h(b' - ha' - b^2 - 2ac + 3abh + 2h^2 - 2a^2h^2 + z).
\end{aligned} \tag{3.14}$$

There are two distinct cases: (i) $d = 0$ and (ii) $d \neq 0$.

Case II.i. $d = 0$. With a proper Möbius transformation, one can set $b = 0, e = 1$. Therefore equations (1.11) and (3.2) become

$$u = \frac{v' + av^2 + c}{v + f} \qquad Av^2 + Bv + C = 0 \tag{3.15}$$

respectively, where

$$\begin{aligned}
&A = -2(a^2 - 1) \qquad B = 3au + b_0 \\
&C = -(u' + u^2 + afu + c_0) \\
&b_0 = a' + 2f(a^2 - 1) \qquad c_0 = fb_0 + 2ac - z.
\end{aligned} \tag{3.16}$$

Then $h = f$ and equation (3.14) imply

$$c = f' - af^2 \qquad f'' = 2f^3 + zf - \alpha. \tag{3.17}$$

When $A \neq 0$, u satisfies the following second-order second degree equation of Painlevé type:

$$\begin{aligned}
&[8(a^2 - 1)^2u'' + 2a(a^2 - 1)(3au - 7a' + 6fa^2 - 6f)u' - Q_3(u)]^2 \\
&\qquad = [2a(a^2 - 1)u' - R_2(u)]^2\Delta
\end{aligned} \tag{3.18}$$

where

$$\Delta = -[8(a^2-1)u' - (a^2+8)u^2 - 2aa_1u - a_0]$$

$$\begin{aligned}
Q_3(u) &= (a^2+8)(a^2+2)u^3 + a[(5a^2-14)a' + 6f(a^2-1)(a^2+4)]u^2 \\
&\quad + [2a(a^2-1)(a_1' + 2afa_1) - 2a_1(2a^2+1)a' + a_0(a^2+2)]u \\
&\quad + (a^2-1)(a_0' + 2afa_0) - 3aa'a_0 \\
R_2(u) &= a(a^2-10)u^2 + 2[(a^2+2)a' + 2f(a^2-1)(a^2-3)]u \\
&\quad - 2(a^2-1)[a'' + 2c(a^2-3) + 2az] + 3aa'^2 \\
a_1 &= 3a' + 2f(a^2-1) \\
a_0 &= a' - 4f(a^2-1)a' - 12f^2(a^2-1)^2 - 8(a^2-1)(2ac-z).
\end{aligned} \tag{3.19}$$

Case II.ii. $d \neq 0$. Without loss of generality we set $a = 0, d = 1$. Then the first equation of (3.14) gives

$$(e-2h)(h' + bh - c) = 0. \tag{3.20}$$

If $e = 2h$, then $f = h^2$ and equations (1.11) and (3.2) become

$$u = \frac{v' + bv + c}{(v+h)^2} \qquad Av^2 + Bv + C = 0 \tag{3.21}$$

respectively, where

$$\begin{aligned}
&A = 2(u^2-1) \qquad B = u' + 4hu^2 - 3bu + 2h \\
&C = hu' + 2h^2u^2 + (a_1 - 3bh)u + a_0 + 4h^2 \\
&a_1 = 2(h' + bh - c) \qquad a_0 = -(b' - b^2 + 6h^2 + z) \\
&c' - bc + \alpha + h(a_0 + 4h^2) = 0.
\end{aligned} \tag{3.22}$$

The discriminant Δ of the second equation of (3.21) is

$$\Delta = (u' - 3bu + 6h)^2 - 8(u^2-1)(a_1u + a_0). \tag{3.23}$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero, then u satisfies the following second degree equation

$$\begin{aligned}
&[4(u^2-1)(a_1u + a_0)u'' - 3(2a_1u^2 + a_0u - a_1)u'^2 - 2F_3(u)u' + Q_4(u)]^2 \\
&\qquad = [3(a_0u + a_1)u' - R_3(u)]^2\Delta
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
F_3(u) &= (a_1' - 3ba_1)u^3 + (a_0' + 3ba_0 + 18ha_1)u^2 - (a_1' - 6ha_0)u - (a_0' + 6ba_0 + 12ha_1) \\
Q_4(u) &= 2[3b(a_1' - 2ba_1) + 2a_1(a_0 + 18h^2 + 3z)]u^4 - [12h(a_1' + 2ba_1) \\
&\quad - 4a_1(a_1 + 6c) - 3b(2a_0' - ba_0) - 4a_0(a_0 + 18h^2 + 3z)]u^3 \\
&\quad - 3[b(2a_1' - 7ba_1) + 4a_1(6h^2 + z) + 4ha_0' + 4a_0(5bh - 2c)]u^2 \\
&\quad + 2[6h(a_1' - ba_1) - 2a_1(a_1 + 6c) - 3ba_0' \\
&\quad - 2a_0(a_0 + 18h^2 + 3z - 3b^2)]u + 4[3ha_0' - a_0(a_1 + 6c - 6bh) + 9h^2a_1] \\
R_3(u) &= 2(a_1' - ba_1)u^3 + (2a_0' + ba_0 + 12ha_1)u^2 \\
&\quad - (2a_1' + ba_1 - 6ha_0)u - 2(a_0' + 2ba_0 + 3ha_1).
\end{aligned} \tag{3.25}$$

If $e \neq 2h$, then $c = h' + bh$ and $f = h(e-h)$ and equations (1.11) and (3.2) become

$$u = \frac{v' + bv + c}{(v+h)(v+e-h)} \qquad Av^2 + Bv + C = 0 \tag{3.26}$$

respectively, where

$$\begin{aligned}
&A = 2(u^2-1) \qquad B = u' + (3e-2h)u^2 - 3bu - 6\\
&C = (e-h)u' + e(e-h)u^2 + (a_1 - 3be + 3bh)u + a_0 - 2e(e-h)\\
&a_1 = g' + bg \qquad a_0 = -(b' - b^2 + z + 2e^2 - 2eh + 2h^2)\\
&h'' - 2h^3 - zh + \alpha = 0 \qquad g = e - 2h.
\end{aligned} \tag{3.27}$$

The discriminant Δ of the second equation of (3.26) is

$$\Delta = [u' - gu^2 - 3bu + 2(2e-h)]^2 - 8(u^2-1)(a_1u + a_0). \tag{3.28}$$

If Δ is not a complete square, that is, a_1 and a_0 are not both zero, then by using the linear transformation $u = py + q$, where $p(z)$ and $q(z)$ are solutions of the following equations

$$p' - 2gpq - 3bp = 0 \qquad q' - gq^2 - 3bq + 2(2e-h) = 0 \tag{3.29}$$

$y(z)$ satisfies the following second degree equation

$$\begin{aligned}
&[F_3(y)y'' - p^4Q_2(y)y'^2 - p^3R_3(y)y' - p^4S_6(y)]^2\\
&\qquad = [T_2(y)y' + G_4(y)]^2[p^2(y' - pgy^2)^2 - 2F_3(y)]
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
&F_3(y) = 4(p^2y^2 + 2pqy + q^2 - 1)(c_1y + c_0)\\
&Q_2(y) = 7pf_3y^2 + (3pf_2 + 5qf_3)y + 3pf_1 - 5q(f_2 - 2qc_1)\\
&R_3(y) = [c_1' + g(3p^3f_2 - 5qc_1)]y^3 + [p(pf_2' + 2p'f_2) + g(4p^3f_1 + 5qc_0)]y^2\\
&\qquad\qquad +[p(pf_1' + 2p'f_1) + 4gp^3f_0]y + p(pf_0' + 2p'f_0)\\
&S_6(y) = g^2p^3f_3y^6 - [2p^2(gf_3' - 2g'f_3) - g^2(3qc_1 - pc_0) + 4p^3f_3^2]y^5\\
&\qquad\qquad -[2p^2(gf_2' - 2g'f_2) + 4f_3(3qc_1 - pc_0) - g^2(2p^3f_1 - 5qc_0)]y^4\\
&\qquad\qquad -[2p^2(gf_1' - 2g'f_1) + 4f_2(3qc_1 - pc_0) + 4f_3(p^3f_1 - 5qc_0)]y^3\\
&\qquad\qquad -[2p^2(gf_0' - 2g'f_0) + 4f_2(p^3f_1 - 5qc_0) + 20qc_1f_1 + 5p^3f_0f_3]y^2\\
&\qquad\qquad -4[2p^3f_1^2 - 5qc_0f_1 + f_0(3qc_1 - pc_0)]y - 4f_0(2p^3f_1 - 5qc_0)\\
&T_2(y) = p[p^3f_3y^2 + (5qc_1 - 3pc_0)y + 11qc_0 - 4p^3f_1]\\
&G_4(y) = gp^5f_3y^4 + p^4[2f_3' - f_3(2gq - b) + 3gpf_2]y^3\\
&\qquad\qquad +p^2[2p^2f_2' + 8f_3(qp' - q'p) + 2gp^3f_1\\
&\qquad\qquad +3gqc_0 + 4p^2f_2(qg + b)]y^2 + [4pqc_0' - 4(p'q + pq')c_0\\
&\qquad\qquad +2(q^2-1)c_1' - 4qq'c_1 + 4gp^5f_0 - 4p^4f_1(gq + b)]y\\
&\qquad\qquad +2[(q^2-1)c_0' - 2qq'c_0 + 2p^4f_0(gq + b)]\\
&c_1 = pa_1 \qquad c_0 = qa_1 + a_0\\
&f_3 = \frac{c_1}{p^2} \qquad f_2 = \frac{2qc_1 + pc_0}{p^3} \qquad f_1 = \frac{c_1(q^2-1) + 2pqc_0}{p^4} \qquad f_0 = \frac{c_0(q^2-1)}{p^4}.
\end{aligned} \tag{3.31}$$

If Δ is a complete square, then $w = \frac{u+1}{2}$ is a solution of PXLV [17, p 342].

4. Painlevé III

Let $v(z)$ be a solution of PIII

$$v'' = \frac{v'^2}{v} - \frac{v'}{z} + \gamma v^3 + \frac{\alpha}{z}v^2 + \frac{\beta}{z} + \frac{\delta}{v}. \tag{4.1}$$

Then, equation (1.13) takes the form of:

$$\begin{aligned}
&[d^2u^2 - 2adu + a^2 - \gamma]v^4 + \left[du' + deu^2 + \left(d' - ae - bd + \frac{d}{z}\right)u\right.\\
&\quad \left. - \left(a' + \frac{a+\alpha}{z} - ab\right)\right]v^3 + \left[eu' + \left(e' + \frac{e}{z}\right)u - \left(b' + \frac{b}{z}\right)\right]v^2\\
&\quad + \left[fu' - efu^2 + \left(f' + \frac{f}{z} + bf + ce\right)u - \left(c' + \frac{c+\beta}{z} + bc\right)\right]v\\
&\quad - [f^2u^2 - 2cfu + c^2 + \delta] = 0.
\end{aligned} \tag{4.2}$$

There are three distinct cases to reduce (4.2) to a quadratic equation in v.

Case I. If $d^2u^2 - 2adu + a^2 - \gamma \neq 0$. Then (4.2) can be written as

$$(v^2 + hv + g)(Av^2 + Bv + C) = 0 \tag{4.3}$$

where

$$\begin{aligned}
A &= d^2u^2 - 2adu + a^2 - \gamma\\
B &= du' + d(e - dh)u^2 + \left(d' + \frac{d}{z} - bd - ae + 2adh\right)u\\
&\quad - \left(a' + \frac{a+\alpha}{z} - ab\right) - h(a^2 - \gamma)\\
C &= (e - dh)u' - d[dg + h(e - dh)]u^2\\
&\quad + \left[e' + \frac{e}{z} - h\left(d' + \frac{d}{z} - bd - ae + 2adh\right) + 2adg\right]u\\
&\quad - \left(b' + \frac{b}{z}\right) + h\left(a' + \frac{a+\alpha}{z} - ab\right) + h^2(a^2 - \gamma) - g(a^2 - \gamma)
\end{aligned} \tag{4.4}$$

and a, b, c, d, e, f, g, h satisfy

$$\begin{aligned}
&g(e - dh) = 0 \qquad h(e - dh) = f - dg\\
&dgh(e - dh) = f^2 - d^2g^2 \qquad dh[dg + h(e - dh)] - dg(e - dh) = ef\\
&g\left[e' + \frac{e}{z} - h\left(d' + \frac{d}{z} - bd - ae + 2adh\right) + 2adg\right] = 2cf\\
&g\left[b' + \frac{b}{z} - h\left(a' + \frac{a+\alpha}{z} - ab\right) - h^2(a^2 - \gamma) + g(a^2 - \gamma)\right] = c^2 + \delta\\
&h\left[e' + \frac{e}{z} - h\left(d' + \frac{d}{z} - bd - ae + 2adh\right) + 2adg\right]\\
&\quad + g\left(d' + \frac{d}{z} - bd - ae + 2adh\right) = f' + \frac{f}{z} + bf + ce\\
&h\left[b' + \frac{b}{z} - h\left(a' + \frac{a+\alpha}{z} - ab\right) - h^2(a^2 - \gamma) + 2g(a^2 - \gamma)\right]\\
&\quad + g\left(a' + \frac{a+\alpha}{z} - ab\right) = \left(c' + \frac{c+\beta}{z} + bc\right).
\end{aligned} \tag{4.5}$$

Hence there are two subcases: (i) $e \neq dh$ and (ii) $e = dh$.

Case I.i. $e \neq dh$. Then (4.5) implies $f = g = h = 0$, and

$$c^2 + \delta = 0 \qquad ce = 0 \qquad \frac{c+\beta}{z} + bc = 0. \tag{4.6}$$

Equation (4.6) gives $c = \beta = \delta = 0$. In the case of $\beta = \delta = 0$, the transformation [11]

$$w = z\left(\frac{v'}{v} + \gamma^{1/2}\right) \qquad v = \frac{w'}{\gamma^{1/2}w + \alpha + \gamma^{1/2}}$$

transforms PIII into

$$w'' = \frac{1}{z}ww' \tag{4.7}$$

which has the first integral $zw' = \frac{1}{2}w^2 + w + k$, $k =$constant.

There are two subcases which should be considered separately: (1) $d = 0$ and (2) $d \neq 0$.

Case I.i-1: $d = 0$. Then without loss of generality one can set $b = 0$ and $e = 1$. With these choices equations (1.11) and (4.2) become

$$u = \frac{v' + av^2}{v} \qquad Av^2 + Bv + C = 0 \tag{4.8}$$

respectively, where

$$\begin{aligned} A &= -z(a^2 - \gamma) \\ B &= zau + za' + a + \alpha \\ C &= -(zu' + u). \end{aligned} \tag{4.9}$$

Note that $A \neq 0$, thus the second-order second degree Painlevé-type equation related with PIII is:

$$[2z^2(a^2-\gamma)^2u'' - F_1(u)u' - Q_3(u)]^2 = [za(a^2-\gamma)u' - R_2(u)]^2\Delta \tag{4.10}$$

where

$$\begin{aligned} \Delta &= -[4z^2(a^2-\gamma)u' - z^2a^2u^2 - 2z(zaa' - a^2 + \alpha a + 2\gamma)u - (za' + a + \alpha)^2] \\ F_1(u) &= z(a^2-\gamma)[z(a^2 - 4\gamma)u + zaa' - 5a^2 - 3\alpha a + 2\gamma] \\ Q_3(u) &= \gamma z^2a^2u^3 + z(\gamma zaa' + a^4 + \alpha a^3 - 2\gamma a^2 + 2\alpha\gamma a + 4\gamma^2)u^2 \\ &\quad + [z^2a(a^2-\gamma)a'' - a^2(za' - a)^2 + \alpha(a^2+\gamma)(za' - a) \\ &\quad + 2(\alpha a + \gamma)^2 + \gamma(2a^2 + 4\alpha a + \alpha^2)]u \\ &\quad + \left(a' + \frac{1}{z}a + \frac{1}{z}\alpha\right)[z^2(a^2-\gamma)a'' \\ &\quad - z^2aa'^2 + z(a^2-\gamma)a' + \alpha(a^2+\gamma) + (\alpha^2+\gamma)a] \\ R_2(u) &= \gamma zau^2 - (a^3 - \alpha a^2 - 3\gamma a - \gamma\alpha)u + (a^2-\gamma)(za'' + 2a') \\ &\quad - (za' + a + \alpha)\left(aa' - \frac{1}{z}\alpha a - \frac{1}{z}\gamma\right). \end{aligned} \tag{4.11}$$

As a special case of (4.10), if $a = 0, \gamma \neq 0$ then the transformation

$$u = e^{-x}\left(y + \frac{\alpha^2}{4\gamma}x\right) \qquad z = e^x \tag{4.12}$$

transforms (4.10) to the following second degree equation

$$\ddot{y} = 2(y + a_0)\dot{y} + 2(b_1 y + b_0)\sqrt{\dot{y}} \tag{4.13}$$

where

$$a_0 = 1 - \frac{\alpha^2}{4\gamma}x \qquad b_1^2 = \frac{\alpha^2}{\gamma} \qquad b_0 = b_1 a_0.$$

Equation (4.13) was also obtained in [2].

Case I.i-2. $d \neq 0$. With a proper Möbius transformation one can set $a = 0$ and $d = 1$. Hence, equations (1.11) and (4.2) respectively become

$$u = \frac{v' + bv}{v^2 + ev} \qquad Av^2 + Bv + C = 0 \tag{4.14}$$

where

$$\begin{aligned} A &= u^2 - \gamma \\ B &= u' + eu^2 - \left(b - \frac{1}{z}\right)u - \frac{1}{z}\alpha \\ C &= eu' + \left(g_1 - be + \frac{1}{z}e\right)u + g_0 - \frac{1}{z}\alpha e + \gamma e^2 \\ g_1 &= e' + be \qquad g_0 = -\left(b' + \frac{b - \alpha e}{z} + \gamma e^2\right). \end{aligned} \tag{4.15}$$

The discriminant Δ of the second equation of (4.14) is

$$\Delta = \left[u' - eu^2 - \left(b - \frac{1}{z}\right)u + 2\gamma e - \frac{1}{z}\alpha\right]^2 - 4(u^2 - \gamma)(g_1 u + g_0). \tag{4.16}$$

If g_1 and g_0 are not both zero, then $y = \frac{u-q}{p}$, where $p(z)$ and $q(z)$ are solutions of the following equations

$$p' - 2epq - p\left(b - \frac{1}{z}\right) = 0 \qquad q' - eq^2 - q\left(b - \frac{1}{z}\right) + 2\gamma e - \frac{1}{z}\alpha = 0 \tag{4.17}$$

satisfies the following second degree Painlevé equation

$$\begin{aligned} &[pF_3(y)y'' - pQ_2(y)y'^2 - R_4(y)y' + p^2 S_5(y)]^2 \\ &\qquad = p^2[T_2(y)y' + G_3(y)]^2[(y' - epy^2)^2 - 2p^2 F_3(y)] \end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
F_3(y) &= 2(c_1y+c_0)(y^2+2a_1y+a_0)\\
Q_2(y) &= 3c_1y^2+(c_0+5a_1c_1)y+a_1c_0+2a_0c_1\\
R_4(y) &= ep^2c_1y^4+[pc_1'+2p'c_1+2ep^2(c_0-a_1c_1)]y^3\\
&\quad +[pf_2'+2p'f_2+ep^2(8a_1c_0+a_0c_1)]y^2\\
&\quad +[pf_1'+2p'f_1+4ep^2f_0]y+pf_0'+2p'f_0\\
S_5(y) &= [ec_1'-2e'c_1+pe^2(c_0-a_1c_1)]y^5\\
&\quad +[ef_2'-2e'f_2-4c_1(c_0-a_1c_1)+pe^2(a_1c_0-a_0c_1)]y^4\\
&\quad +[ef_1'-2e'f_1-4pc_1(a_1c_0-a_0c_1)-4pf_2(c_0-a_1c_1)]y^3\\
&\quad +[ef_0'-2e'f_0-4pf_2(a_1c_0-a_0c_1)-4pf_1(c_0-a_1c_1]y^2\\
&\quad -4p[f_1(a_1c_0-a_0c_1)+f_0(c_0-a_1c_1)]y-4pf_0(a_1c_0-a_0c_1)\\
T_2(y) &= c_1y^2-(c_0-3a_1c_1)y+2a_0c_1-a_1c_0\\
G_3(y) &= [c_1'+c_1(eq-2b)+epf_2]y^3\\
&\quad +[f_2'-4c_1a_1'+ep(3a_1c_0+a_0c_1)+2f_2(eq-b)]y^2\\
&\quad +[f_1'-2c_1a_0'-4c_0a_1'+2epf_0+2f_1(eq-b)]y\\
&\quad +f_0'-2c_0a_0'+2f_0(eq-b)\\
a_1 &= \frac{1}{p}q \qquad a_0=\frac{1}{p^2}(q^2-\gamma) \qquad c_1=\frac{1}{p}g_1 \qquad c_0=\frac{1}{p^2}(qg_1+g_0)\\
f_2 &= 2a_1c_1+c_0 \qquad f_1=2a_1c_0+a_0c_1 \qquad f_0=a_0c_0.
\end{aligned} \tag{4.19}$$

If $g_1=g_0=0$ and $\gamma\neq 0$ then $w=\frac{u-\sqrt{\gamma}}{u+\sqrt{\gamma}}$ is a solution of PXL in [17, p 341]. If $g_1=g_0=0$ and $\gamma=0$ then $w=\frac{\alpha}{u}$ is a solution of PXVI in [17, p 335]. It should be noted that both PXL and PXVI have first integrals [17].

Case I.ii: $e=dh$. Then the second equation of (4.5) gives $f=dg$ and hence $d\neq 0$. Without loss of generality one can take $a=0$ and $d=1$. Thus (4.4) and (4.5) yield respectively

$$\begin{aligned}
A &= u^2-\gamma\\
B &= u'-\left(b-\frac{1}{z}\right)u+\gamma e-\frac{1}{z}\alpha\\
C &= -(fu^2+g_1u+g_0)\\
g_1 &= -(e'+be) \qquad g_0=b'+\frac{1}{z}b-\frac{\alpha}{z}e+\gamma e^2-\gamma f
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&f(g_1+2c)=0 \qquad e(g_1+c)+f'+2bf=0\\
&fg_0=c^2+\delta \qquad eg_0=f\left(\gamma e-\frac{1}{z}\alpha\right)+c'+\frac{c+\beta}{z}+bc=0.
\end{aligned} \tag{4.21}$$

Thus (1.11) becomes

$$u=\frac{v'+bv+c}{v^2+ev+f}. \tag{4.22}$$

The discriminant Δ of $Av^2 + Bv + C = 0$ is

$$\Delta = \left[u' - \left(b - \frac{1}{z}\right)u + \gamma e - \frac{1}{z}\alpha\right]^2 + 4(u^2 - \gamma)(fu^2 + g_1 u + g_0). \tag{4.23}$$

If Δ is not a complete square, then one obtains the following the second-order second degree equation related with PIII

$$\begin{aligned}&[2(u^2-\gamma)(cu-g_0)u'' - u(cu-g_0)u'^2 + F_3(u)u' + Q_5(u)]^2 \\ &\qquad = [u(cu-g_0)u' + R_4(u)]^2\Delta\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
F_3(u) &= (u^2-\gamma)(3cb_1u + g_0' - 2b_1g_0 + cb_0)\\
Q_5(u) &= 8\delta u^5 + 2c(b_1' + b_1^2 + 6g_0)u^4\\
&\quad -(2g_0b_1' - 2cb_0' - b_1g_0' + 4g_0^2 + g_0b_1^2 - 4cb_0b_1 - 12\gamma\delta)u^3\\
&\quad -(2g_0b_0' + 2\gamma cb_1' - b_0g_0' + 16\gamma cg_0 + \gamma cb_1^2 + 2g_0b_0b_1 - 2cb_0^2)u^2\\
&\quad +(2\gamma g_0b_1' - 2c\gamma b_0' - \gamma b_1g_0' + 4\gamma g_0^2 - 2c\gamma b_0b_1 + 4\delta\gamma^2 - g_0b_0^2)u\\
&\quad +\gamma(2g_0b_0' + b_0g_0' + 4c\gamma g_0 - cb_0^2)\\
R_4(u) &= 2ceu^4 - 2(eg_0 + bc)u^3 + (g_0' + 2bg_0 + b_1g_0 - 2\gamma ce)u^2\\
&\quad +(2cb + 2\gamma eg_0 - \gamma cb_1 - b_0g_0)u - \gamma(g_0' + 2bg_0 + cb_0)\\
b_1 &= -\left(b - \frac{1}{z}\right) \qquad b_0 = \gamma e - \frac{1}{z}\alpha
\end{aligned} \tag{4.25}$$

if $f = 0$. If $f \neq 0$ then by using the linear transformation $u = py + q$, where $p(z)$ and $q(z)$ are given as follows

$$p(z) = \frac{1}{z}\exp\left(\int^z b(s)\,\mathrm{d}s\right) \qquad q(z) = -p(z)\int^z \frac{1}{p(s)}\left[\gamma e(s) - \frac{\alpha}{s}\right]\mathrm{d}s \tag{4.26}$$

$y(z)$ solves the following second degree equation of Painlevé type

$$\begin{aligned}&[F_4(y)y'' - Q_3(y)y'^2 - R_4(y)y' + F_4(y)S_3(y)]^2 \\ &\qquad = p^2[T_2(y)y' - G_5(y)]^2[y'^2 + 2pF_4(y)]\end{aligned} \tag{4.27}$$

where

$$
\begin{aligned}
F_4(y) &= 2p(y^2+2a_1y+a_0)(fy^2+2c_1y+c_0)\\
Q_3(y) &= p(2fy^3+3f_3y^2+f_2y+f_1)\\
R_4(y) &= (pf'+2p'f)y^4+(pf_3'+2p'f_3)y^3\\
&\quad+(pf_2'+2p'f_2)y^2+(pf_1'+2p'f_1)y+(pf_0'+2p'f_0)\\
S_3(y) &= 2p^3[2fy^3+3f_3y^2+f_2y+f_1]\\
T_2(y) &= \frac{c}{p}y^2+(f_2-2c_0-4a_1c_1)y+f_1-2a_1c_0\\
G_5(y) &= 2epfy^5-[f'-2f(eq-b)-4epf_3]y^4\\
&\quad-2[f_3'-2fa_1'-2f_3(eq-b)-epf_2]y^3\\
&\quad-[f_2'-8c_1a_1'-2fa_0'-2f_2(eq-b)-4epf_1]y^2\\
&\quad-2[f_1'-2c_0a_1'-2c_1a_0'-2f_1(eq-b)-epf_0]y\\
&\quad-[f_0'-2c_0a_0'-2f_0(eq-b)]
\end{aligned}
\tag{4.28}
$$

$$
a_0=\frac{1}{p^2}(q^2-\gamma)\qquad a_1=\frac{q}{p}\qquad c_1=\frac{1}{p}(fq-c)\qquad c_0=\frac{1}{p^2f}(p^2c_1^2+\delta)
$$

$$
f_3=c_1+fa_1\qquad f_2=c_0+4a_1c_1+fa_0\qquad f_1=a_1c_0+a_0c_1\qquad f_0=a_0c_0.
$$

If Δ is a complete square, that is, $C=0$, then $ec=0$. If $e\neq 0$ then this case reduces to the case I.i-2 with $a_1=a_0=0$. If $e=0$ and $\gamma=0$, then $w=zu$ is a solution of PIII. If $e=0$ and $\gamma\neq 0$, then $w(x)=\frac{u-\sqrt{\gamma}}{u+\sqrt{\gamma}}$, where $z^2=2x$, is a solution of PV with $\delta=0$ [11].

Case II. $d^2u^2-2adu+a^2-\gamma=0$ $du'+deu^2+(d'-ae-bd+\frac{d}{z})u-(a'+\frac{a+\alpha}{z}-ab)\neq 0.$ Then (4.2) can be written as:

$$
(v+f)(Av^2+Bv+C)=0 \tag{4.29}
$$

where

$$
\begin{aligned}
A &= au+\frac{a+\alpha}{z}\\
B &= -\left[u'+\left(af+\frac{1}{z}\right)u+f\left(\frac{a+\alpha}{z}\right)\right]\\
C &= fu^2-(f'-af^2+c)+\left(c'+\frac{c+\beta}{z}+\frac{a+\alpha}{z}f^2\right)
\end{aligned}
\tag{4.30}
$$

and

$$
\begin{aligned}
&a^2-\gamma=0\qquad f(f'-af^2-c)=0\\
&f\left(c'+\frac{c+\beta}{z}+\frac{a+\alpha}{z}f^2\right)=c^2+\delta.
\end{aligned}
\tag{4.31}
$$

If $f\neq 0$ then (4.31) implies

$$
c=f'-af^2\qquad c'+\frac{c+\beta}{z}+\frac{a+\alpha}{z}f^2=\frac{1}{f}(c^2+\delta). \tag{4.32}
$$

The discriminant Δ of $Av^2+Bv+C=0$ reads

$$
\Delta=\left[u'+\left(af+\frac{1}{z}\right)u+f\left(\frac{a+\alpha}{z}\right)\right]^2-4\left(au+\frac{a+\alpha}{z}\right)\left(fu^2-2cu+\frac{c^2+\delta}{f}\right) \tag{4.33}
$$

and equation (1.11) becomes

$$u = \frac{v' + av^2 + c}{v + f}. \tag{4.34}$$

Let $y = \frac{u-q}{p}$, where $p(z)$ and $q(z)$ are given as

$$p(z) = \frac{1}{z}\exp\left[-a\int^z f(s)\,\mathrm{d}s\right] \tag{4.35}$$

$$\begin{aligned} q(z) &= \frac{a+\alpha}{a}\left[p(z) - \frac{1}{z}\right] \qquad a \neq 0 \\ q(z) &= \frac{-\alpha}{z}\int^z f(s)\,\mathrm{d}s \qquad a = 0. \end{aligned} \tag{4.36}$$

Then $y(z)$ is a solution of the following second degree Painlevé-type equation

$$[F_3(y)y'' - Q_2(y)y'^2 - R_3(y)y' - F_3(y)S_2(y)]^2 = [Q_2(y)y' - T_4(y)]^2[y'^2 - 2pF_3(y)] \tag{4.37}$$

where

$$\begin{aligned} F_3(y) &= 2(ay+\sigma)(fy^2 + 2c_1y + c_0) \\ Q_2(y) &= (ay+\sigma)(fy+c_1) \\ R_3(y) &= a\left(f' + \frac{p'}{p}f\right)y^3 + \left(f_2' + \frac{p'}{p}f_2\right)y^2 + \left(f_1' + \frac{p'}{p}f_2\right)y + \sigma\left(c_0' + \frac{p'}{p}c_0\right) \\ S_2(y) &= 2p[2afy^2 + (3ac_1 + \sigma f)y + ac_0 + \sigma c_1] \\ T_4(y) &= 2apfy^4 - \left[a\left(f' + \frac{p'}{p}f\right) - 2(pf_2 + aqf)\right]y^3 \\ &\quad - \left[\sigma\left(f' + \frac{p'}{p}f\right) + 2a\left(c_1' + \frac{p'}{p}c_1\right) - 2(pf_1 + qf_2)\right]y^2 \\ &\quad - \left[2\sigma\left(c_1' + \frac{p'}{p}c_1\right) + a\left(c_0' + \frac{p'}{p}c_0\right) - 2(\sigma pc_0 + qf_1)\right]y \\ &\quad - \left[\sigma\left(c_0' + \frac{p'}{p}c_0\right) - 2\sigma qc_0\right] \\ \sigma &= a + \alpha \qquad c_1 = \frac{1}{p}(fq - c) \qquad c_0 = \frac{1}{p^2 f}(p^2c_1^2 + \delta) \\ f_2 &= \sigma f + 2ac_1 \qquad f_1 = ac_0 + 2\sigma c_1. \end{aligned} \tag{4.38}$$

If $f = 0$, then $w = zu$ is a solution of the equation

$$\begin{aligned} &\left(w'' + \rho w + \frac{\sigma}{2}\right)^2 = \frac{w^2}{z^2}(w'^2 + \rho w^2 + \sigma w + \tau) \\ &\rho = 4\gamma^{1/2}(-\delta)^{1/2} \qquad \sigma = 4[\alpha(-\delta)^{1/2} - \beta\gamma^{1/2}] \\ &\tau = -4(\alpha + \gamma^{1/2})[\beta + (-\delta)^{1/2}]. \end{aligned} \tag{4.39}$$

Equation (4.39) was first obtained in [11] and also in [8] which was denoted as SD-III$'$. When $f = c = 0$ then Δ is a complete square, and $w = zu$ is a solution of (4.7).

Case III. $d^2u^2 - 2adu + a^2 - \gamma = 0$ *and* $du' + deu^2 + \left(d' - ae - bd + \frac{d}{z}\right)u - \left(a' + \frac{a+\alpha}{z} - ab\right) = 0$. Then (1.11) and (4.2) become

$$u = \frac{v' + av^2 + bv + c}{ev + f} \qquad Av^2 + Bv + C = 0 \tag{4.40}$$

respectively, where

$$\begin{aligned} A &= eu' + \left(e' + \frac{1}{z}e\right)u - \left(b' + \frac{1}{z}b\right) \\ B &= fu' - efu^2 + \left(f' + \frac{1}{z}f + bf + ce\right)u - \left(c' + \frac{c+\beta}{z} + bc\right) \\ C &= -(f^2u^2 - 2cfu + c^2 + \delta). \end{aligned} \tag{4.41}$$

and

$$a^2 - \gamma = 0 \qquad ae = 0 \qquad \frac{a+\alpha}{z} - ab = 0. \tag{4.42}$$

The discrete Lie-point symmetry of PIII [11]

$$\bar{v} = \frac{1}{v} \qquad \bar{\alpha} = \beta \qquad \bar{\beta} = \alpha \qquad \bar{\gamma} = -\delta \qquad \bar{\delta} = -\gamma \tag{4.43}$$

transforms this case to the case I.i.

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